Short Stickelberger Class Relations and application to Ideal-SVP

Ronald Cramer Léo Ducas Benjamin Wesolowski

Leiden University, The Netherlands

CWI, Amsterdam, The Netherlands

EPFL, Lausanne, Switzerland

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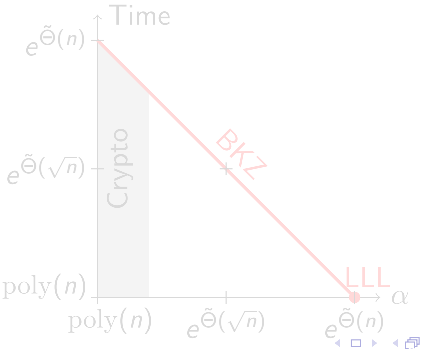
Lattice-Based Crypto

Lattice problems provides a strong fundation for Post-Quantum Crypto

Worst-case to average-case reduction [Ajtai, 1999, Regev, 2009]

$$\text{Worst-case Approx-SVP} \geq \begin{cases} \text{SIS} & \text{(Short Intreger Solution)} \\ \text{LWE} & \text{(Learning With Error)} \end{cases}$$

How hard is Approx-SVP ? Depends on the Approximation factor α.



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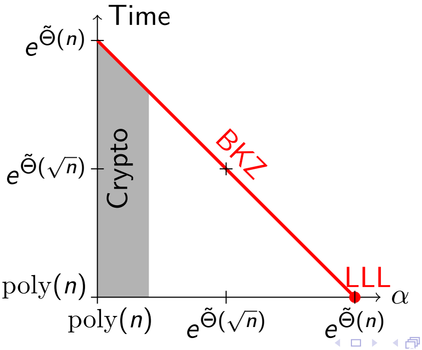

Lattices over Rings (Ideals, Modules)

Generic lattices are cumbersome! Key-size $= \tilde{O}(n^2)$.

NTRU Cryptosystems [Hoffstein et al., 1998, Hoffstein et al., 2003]

Use the convolution ring $\mathcal{R} = R[X]/(X^p - 1)$, and module-lattices:

$$\mathcal{L}_h = \{(x, y) \in \mathcal{R}^2, \quad hx + y \equiv 0 \bmod q\}.$$

Same lattice dimension, Key-Size $= \tilde{O}(n)$. Later came variants with worst-case fundations:

wc-to-ac reduction [Micciancio, 2007, Lyubashevsky et al., 2013]

$$\text{Worst-case Approx-Ideal-SVP} \geq \begin{cases} \text{Ring-SIS} \\ \text{Ring-LWE} \end{cases}$$

Applicable for cyclotomic rings $\mathcal{R} = \mathbb{Z}[\omega_m]$ (ω_m a primitive m-th root of unity).

Denote $n = \deg \mathcal{R}$. In our cyclotomic cases: $n = \phi(m) \sim m$.

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Is Ideal-SVP as hard as general SVP ?

Are there other approach than lattice reduction (LLL,BKZ) ?
An algebraic approach was sketched in [Campbell et al., 2014]:

The Principal Ideal Problem (PIP)

Given a **principal ideal** $\mathfrak{h}$, recover a generator h s.t. $h\mathcal{R} = \mathfrak{h}$.

Solvable in quantum poly-time [Biasse and Song, 2016].

The Short Generator Problem (SGP)

Given a generator h, recover another **short** generator g s.t. $g\mathcal{R} = h\mathcal{R}$.

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Are Ideal-SVP and Ring-LWE broken ?!

Not quite yet ! 3 serious obstacle remains:

(i) Restricted to **principal** ideals.

(ii) The approximation factor in **too large** to affect Crypto.

(iii) Ring-LWE $\geq$ Ideal-SVP, but **equivalence is not known**.

Approaches ?

(i) Solving the **Close Principal Multiple** problem (CPM) [**This work !**]

(ii) Considering many CPM solutions [Plausible]

(iii) Generalization of LLL to **non-euclidean** rings [Seems tough]

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Our result: Ideal-SVP in poly-time for large α

This work: CPM via Stickelberger Short Class Relation

$\Rightarrow$ Ideal-SVP **solvable** in Quantum poly-time, for

$$\mathcal{R} = \mathbb{Z}[\omega_m], \quad \alpha = \exp(\tilde{O}(\sqrt{n})).$$

Better tradeoffs

Impact and limitations

- No schemes broken
- **Hardness gap** between SVP and Ideal-SVP
- New cryptanalytic tools

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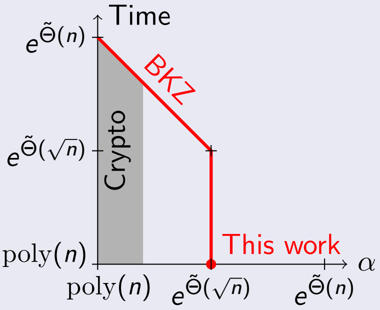

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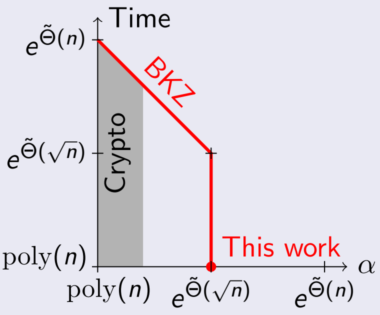

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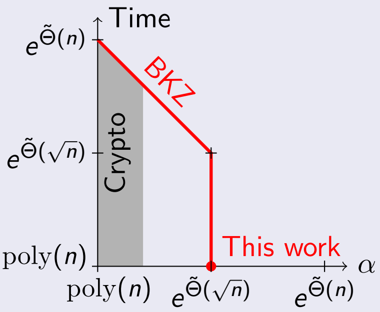

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Table of Contents

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Ideals and Principal Ideals

Cyclotomic number field: $K (= \mathbb{Q}(\omega_m))$, ring of integer $\mathcal{O}_K (= \mathbb{Z}[\omega_m])$.

Definition (Ideals)

- An **integral ideal** is a subset $\mathfrak{h} \subset \mathcal{O}_K$ closed under addition, and by multiplication by elements of $\mathcal{O}_K$,
- A **(fractional) ideal** is a subset $\mathfrak{f} \subset K$ of the form $\mathfrak{f} = \frac{1}{x}\mathfrak{h}$, where $x \in \mathbb{Z}$,
- A **principal ideal** is an ideal $\mathfrak{f}$ of the form $\mathfrak{f} = g\mathcal{O}_K$ for some $g \in K$.

In particular, ideals are lattices.

We denote $\mathcal{F}_K$ the set of fractional ideal,
and $\mathcal{P}_K$ the set of principal ideals.

Class Group

Ideals can be multiplied, and remain ideals:

$$\mathfrak{a}\mathfrak{b} = \left\{ \sum_{\text{finite}} a_i b_i, \quad a_i \in \mathfrak{a}, b_i \in \mathfrak{b} \right\}.$$

The product of two principal ideals remains principal:

$$(a\mathcal{O}_K)(b\mathcal{O}_K) = (ab)\mathcal{O}_K.$$

$\mathcal{F}_K$ form an **abelian group**[1], $\mathcal{P}_K$ is a **subgroup** of it.

Definition (Class Group)

Their quotient form the **class group** $\mathrm{Cl}_K = \mathcal{F}_K / \mathcal{P}_K$.
The class of a ideal $\mathfrak{a} \in \mathcal{F}_K$ is denoted $[\mathfrak{a}] \in \mathrm{Cl}_K$.

An ideal $\mathfrak{a}$ is principal iff $[\mathfrak{a}] = [\mathcal{O}_K]$.

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From CPM to Ideal-SVP

Definition (The Close Principal Multiple problem)

- Given an ideal $\mathfrak{a}$, and an factor F
- Find a **small integral** ideal $\mathfrak{b}$ such that $[\mathfrak{a}\mathfrak{b}] = [\mathcal{O}_K]$ and $N\mathfrak{b} \leq F$

Note: Smallness with respect to the Algebraic Norm N of $\mathfrak{b}$,
(essentially the **volume** of $\mathfrak{b}$ as a lattice).

- Solve CPM, and apply the previous results (PIP-SGP) to $\mathfrak{a}\mathfrak{b}$
- This will give a generator g of $\mathfrak{a}\mathfrak{b} \subset \mathfrak{a}$ (so $g \in \mathfrak{a}$) of length

$$L = N(\mathfrak{a}\mathfrak{b})^{1/n} \cdot \exp(\tilde{O}(\sqrt{n}))$$

- This Ideal-SVP solution has an approx factor of

$$\alpha \approx L/N(\mathfrak{a}) = F^{1/n} \cdot \exp(\tilde{O}(\sqrt{n}))$$

CPM with $F = \exp(\tilde{O}(n^{3/2}))$ $\Rightarrow$ Ideal-SVP with $\alpha = \exp(\tilde{O}(\sqrt{n}))$

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Factor Basis, Class-Group Discrete-Log

Choose a **factor basis** $\mathfrak{B}$ of integral ideals and search $\mathfrak{b}$ of the form:

$$\mathfrak{b} = \prod_{\mathfrak{p} \in \mathfrak{B}} \mathfrak{p}^{e_\mathfrak{p}}.$$

Theorem (Quantum Cl-DL, Corollary of [Biasse and Song, 2016])

Assume $\mathfrak{B}$ generates the class-group. Given $\mathfrak{a}$ and $\mathfrak{B}$, one can find in quantum polynomial time a vector $\vec{e} \in \mathbb{Z}^{\mathfrak{B}}$ such that:

$$\prod_{\mathfrak{p} \in \mathfrak{B}} \left[\mathfrak{p}^{e_\mathfrak{p}}\right] = \left[\mathfrak{a}^{-1}\right].$$

This finds a $\mathfrak{b}$ such that $[\mathfrak{a}\mathfrak{b}] = [\mathcal{O}_K]$, yet:

- $\mathfrak{b}$ may not be integral (negative exponents, yet easy to solve)
- $N\mathfrak{b} \approx \exp(\|\vec{e}\|_1)$ may be huge (unbounded $\vec{e}$, want $\|\vec{e}\|_1 = \tilde{O}(n^{3/2})$).

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Navigating the Class-Group

Cayley-Graph(G, A):

- A node for any element $g \in G$
- An arrow $g \xrightarrow{a} ga$ for any $g \in G$, $a \in A$

Figure: Cayley-Graph$((\mathbb{Z}/5\mathbb{Z}, +)$,$\{1$,$2\})$

Rephrased Goal for CPM

Find a **short** path from $[\mathfrak{a}]$ to $[\mathcal{O}_K]$ in Cayley-Graph$(\mathrm{Cl}, \mathfrak{B})$.

- Using a few well chosen ideals in $\mathfrak{B}$, Cayley-Graph$(\mathrm{Cl}, \mathfrak{B})$ is an **expander Graph** [Jetchev and Wesolowski, 2015]: very short path exists.
- Finding such short path generically too costly: $|\mathrm{Cl}| > \exp(n)$

A lattice problem

Cl is **abelian** and **finite**, so $\mathrm{Cl} = \mathbb{Z}^{\mathfrak{B}}/\Lambda$ for some lattice Λ:

$$\Lambda = \left\{ \vec{e} \in \mathbb{Z}^{\mathfrak{B}}, \quad s.t. \prod [\mathfrak{p}_{\mathfrak{p}}^{e}] = [\mathcal{O}_K] \right\}$$

i.e. the (full-rank) **lattice of class-relations** in base $\mathfrak{B}$.

Figure: $(\mathbb{Z}/5\mathbb{Z}, +) = \mathbb{Z}^{\{1,2\}}/\Lambda$

Rephrased Goal for CPM: CVP in Λ

Find a **short** path from $t \in \mathbb{Z}^{\mathfrak{B}}$ to any lattice point $v \in \Lambda$.

In general: very hard. But for good Λ, with a good basis, can be easy.

Why should we know anything special about Λ ?

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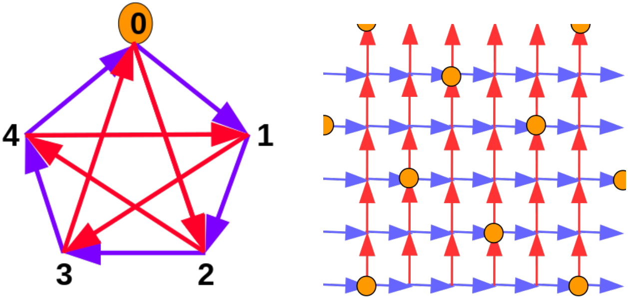
Example

Figure: Cayley-Graph$(\mathbb{Z}/5\mathbb{Z}, \{1, 2\}) \quad \simeq \quad \mathbb{Z}^{\{1,2\}}/\Lambda$



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More than just a lattice

Let G denote the Galois group, it acts on ideals and therefore on classes:

$$[\mathfrak{a}]^{\sigma} = [\sigma(\mathfrak{a})].$$

Consider the **group-ring** $\mathbb{Z}[G]$ (formal sums on G), extend the G-action:

$$[\mathfrak{a}]^{e} = \prod_{\sigma \in G} [\sigma(\mathfrak{a})]^{e_\sigma} \qquad \text{where } e = \sum e_\sigma \sigma.$$

- Assume $\mathfrak{B} = \{\mathfrak{p}^{\sigma}, \sigma \in G\}$
- G acts on $\mathfrak{B}$, and so it acts on $\mathbb{Z}^{\mathfrak{B}}$ by permuting coordinates
- the lattice $\Lambda \subset \mathbb{Z}^{\mathfrak{B}}$ is **invariant** by the action of G !

i.e. Λ admits G as a group of **symmetries**

Λ is more than just a lattice: it is a $\mathbb{Z}[G]$-**module**

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Stickelberger's Theorem

In fact, we know much more about Λ !

Definition (The Stickelberger ideal)

The **Stickelberger element** $\theta \in \mathbb{Q}[G]$ is defined as

$$\theta = \sum_{a \in (\mathbb{Z}/m\mathbb{Z})^*} \left(\frac{a}{m} \bmod 1 \right) \sigma_a^{-1} \qquad \text{where } G \ni \sigma_a : \omega \mapsto \omega^a.$$

The **Stickelberger ideal** is defined as $S = \mathbb{Z}[G] \cap \theta\mathbb{Z}[G]$.

Theorem (Stickelberger's theorem [Washington, 2012, Thm. 6.10])

The Stickelberger ideal annihilates the class group: $\forall e \in S, \mathfrak{a} \subset K$

$$[\mathfrak{a}^e] = [\mathcal{O}_K].$$

In particular, if $\mathfrak{B} = \{\mathfrak{p}^\sigma, \sigma \in G\}$*, then* $S \subset \Lambda$.

Geometry of the Stickelberger ideal

Fact

There exists an **explicit** *(efficiently computable)* **short** *basis of S, precisely it has binary coefficients.*

Corollary

Given $t \in \mathbb{Z}[G]$, one ca find $x \in S$ suh that $\|x - t\|_1 \leq n^{3/2}$.

Conclusion: back to CPM

The CPM problem can be solved with approx. factor $F = \exp(\tilde{O}(n^{3/2}))$.
QED.

Extra technicalities

Convenient simplifications/omissions made so far:

$\mathfrak{B} = \{\mathfrak{p}^\sigma, \sigma \in G\}$ generates the class group.

- can allow a few (say polylog) many different ideals and their conjugates in $\mathfrak{B}$
- Numerical computation says such $\mathfrak{B}$ it should exists [Schoof, 1998]
- Theorem+Heuristic then says we can find such $\mathfrak{B}$ efficiently

Eliminating minus exponents

- Easy when $h^+ = 1$: $[\mathfrak{a}^{-1}] = [\bar{\mathfrak{a}}]$, doable when $h^+ = \mathrm{poly}(n)$
  h^+ is the size of the class group of K^+, the maximal totally real subfield of K
- $h^+ = \mathrm{poly}(n)$ already needed for previous result [Cramer et al., 2016]
- Justified by numerical computations and heuristics [Buhler et al., 2004, Schoof, 2003]

Open questions

Obstacle toward attacks Ring-LWE

(i) Restricted to **principal** ideals.

(ii) The approximation factor in **too large** to affect Crypto.

(iii) Ring-LWE $\geq$ Ideal-SVP, but **equivalence is not known**.

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